Fuzzy Portfolio Optimization Using Tracking Error

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ABSTRACT

The study compares the conventional mean-variance model with the fuzzy mean-variance model for better portfolio construction. Trapezoidal fuzzy number for fuzzy mean, variance, and covariance is computed. The models are applied to BRICS (Brazil, Russia, India, China, and South Africa). For comparison of two alternative models, efficient frontiers are constructed, and tracking error by using the MSCI emerging market index as a benchmark index is measured. The construction is based on both equally weighted and optimized portfolios. The results show that the fuzzy mean-variance model is a better approach as compared to the simple M-V model for BRICS in efficient frontiers settings. Moreover, it suggests that the fuzzy M-V model can help more in devising an investment strategy based on the risk-return relationship.

Keywords: *M-V model, Fuzzy M-V model, Trapezoidal Fuzzy Number, Tracking error.*

INTRODUCTION

The financial market requires advanced strategies and tools for shareholders to increase returns during risk management. The technique of reducing risk by investing in different financial instruments, markets, or securities is called diversification, first developed by Harry Markowitz in 1952 based on the M-V model (Varghese & Joseph, 2018). Despite its wide usage, there are some limitations as explained by Mohamed et al. (2010a), which are crucial to the assumption that future returns will be followed by historical information. Conventional portfolio selection methods assume that historical data accurately predicts the future conditions of the stock market. Whereas, in real real-time situation, historical data is sensitive to uncertain events (Mohamed et al., 2010).

In order to deal with the uncertainty of the stock market, fuzzy logic, proposed by Lotfi Zadeh in 1965, plays a vital role in dealing with vague conditions of portfolio selection (Glensk & Madlener, 2018). Portfolio selection problem moves towards taking fuzzy variables as returns rather than conventional random variables (Liagkouras & Metaxiotis, 2018). As the theory deals with human behavior and thinking aspect, it seems appropriate to use fuzzy logic for portfolio selection and optimization. These fuzzy, vague events in portfolio selection can be measured by various methods such as possibility, necessity, and credibility(Liagkouras & Metaxiotis, 2018; Varghese & Joseph, 2018).

Many researchers take fuzziness into account for the portfolio selection problem and optimum portfolio (Glensk & Madlener, 2018; Liagkouras & Metaxiotis, 2018). Ramli & Jaaman (2017) also take into account the concept of expected fuzzy return and fuzzy variance. The conventional portfolio optimization models explained the uncertainties in financial markets through probability theory, but they left some uncertainties to be addressed with the same probability theory. Therefore, the concept of fuzzy theory is employed in order to deal with remaining uncertainties as it deals with problems in the most natural way (Liagkouras & Metaxiotis, 2018). Real situations are usually vague, so fuzzy theory is usually considered a better measure for such situations (Glensk & Madlener, 2018). Possibility theory is an extension of fuzzy logic, which was introduced and progressive simultaneously by Tanaka et al. (2000).

Previous researchers introduced and worked on different fuzzy portfolio selection models as discussed by (Gharakhani & Sadjadi, 2013; Long et al., 2020; Maraj & Kuka, 2019; Shang & Hossen, 2013). The variance model gives the best results when the conditions are suitable and minimum uncertainty exists in the market. Which do not give accurate or reliable results when applied traditional M-V model. Under these circumstances or to deal with this problem, Fuzzy M-V is proposed as an extension to Markowitz Mean-Variance model. The efficacy of the proposed model is tested and justified by comparing with the conventional Markowitz M-V model. This study sticks to the concept of fuzzy membership function and has taken fuzzy trapezoidal numbers for mean-variance and covariance computations.

To Analyse and investigate the results, data from BRICS and MSCI emerging market indices were taken for tracking error computation. This methodology and comparison with conventional method may help explore the methods in more depth. Therefore, this study comprises of following research questions: firstly, is the fuzzy-based M-V model a better measure for portfolio selection as compared to the conventional M-V model for equally weighted portfolio using tracking error? Secondly, is a fuzzy-based mean-variance model a better measure for portfolio selection as compared to the conventional M-V model for optimized portfolio using tracking error?

This study contributes both theoretically and methodologically to the literature. It expands the theory or M-V model by introducing the concept of fuzzy portfolio selection model. Methodologically, portfolios are formed, and the concept of tracking error is incorporated in the model. Practically, this study helps the investors in decision making regarding resource allocation in portfolios. This way, they can form better-performing portfolios providing high returns and less risk. Overall, uncertainty can be cater in a better way employing Fuzzy M-V model. An attractive destination for investors in the coming years, as they hold the potential to offer high returns, and this potential can even be maximized by using the Fuzzy approach. It will also contribute in providing investors and academicians with a more efficient method for portfolio selection and optimization.

LITERATURE REVIEW

Portfolio selection is one of the most important issues in financial markets. A large number of studies are dealing with mean-variance portfolio that provide best allocation of wealth by selecting portfolio incorporate investors preferences on risk and expectation of returns in order to achieve maximum expected profits (Markowitz,1952; Hou & Xu, 2016; Wang et al.,2024). The main contribution of Markowitz's work is the introduction of quantities and scientific approaches to risk management and analysis(Santos-Alamillos et al., 2017). Therefore, the mean-variance portfolio selection framework has been one of the most predominant investment decision rules in financial portfolio selection theory.

Markowitz portfolio theory proposes minimization of portfolio's risk based on the estimated return and risk of individual assets. This framework is concerned with the allocation of wealth among a variety of financial securities so as to achieve a trade-off between the return and risk at the end of the investment horizon (Wang & Wei, 2020). Investment in the stock market would be considered challenging. Before making any decision for investment, an investor deals with random, vague, and stock price ambiguity. The uncertain issues are resolved by new robust model in real situations. A new fuzzy portfolio selection model is made by semi-variance as a risk measured integrations with investors' judgment for future performance (Wang et al.,2011).

Markowitz's mean-variance model in portfolio selection is processed, and assumptions are made for past data trends. The approach of Li and Xu (2015) is extended, and a fuzzy portfolio model is proposed to measure risk and investors' judgment for future performance for handling of uncertain environment. The historical data is experienced and skilled in solving the problem of normality, skewness, and the investor's real position. Downside risk is measured as a semi-variance to meet the investor behavior. To make the selection process more robust, linear programming is used to optimize the result (Chen et al., 2022).

In the Malaysian stock market, portfolio diversification is extended by the mean-variance model using a fuzzy approach.In this study, the efficient portfolios are derived from linear programming optimization tool and measured by efficient frontier index. Extended

M-V had maximum portfolio diversification, benefited the Malaysian stock market as compared to conventional M-V and VBS fuzzy models. Stock market fluctuations became unpredictable and random, and investors cautiously monitored the stock market (Mohamed et al., 2009).

In 1997-1998, the past Asian economic crises and sub-prime problems in the USA and Europe in 2008-09 caused great loss to public sector investors. Portfolio investment as considered an uncertain investment by right combination of asset and correct asset allocation with diversified unsystematic risk. For example, Zulkifli et al. (2010) are unable to outperform market benchmark. Only 32% mutual funds outperformed DJIA in USA market. In 1968, Jenson considered that the operating expenses of mutual funds are unable to beat a buy-and-hold strategy, and portfolio selection strategy is improved. Many portfolios are examined, and a robust model gave maximized benefits and portfolio diversification (Haung et al., 2022).

Fuzzy possibilities and possibility distributions are two kinds of selection models (Messaoudi, 2024). They depend on possible grades of security data. In real world, uncertainty analysis is attained by a multivariate data analysis tool. It showed uncertainty as a probability phenomenon, and possibility data analysis is measured as a possibility phenomenon. In possibility theory, possibility distributions are presented as normal convex fuzzy sets, L-R fuzzy numbers, quadratic and exponential functions. The first portfolio selection model is the natural extension of the Markowitz model for extending possibility as fuzzy probability. Second portfolio selection model is probability distributions. These models are used for data analysis in different ways. According to experts, possibility portfolio model is used in real investment environment. Fuzzy probability method solved the problem of portfolio selection. The selected portfolios minimize the variance return of the portfolio in the fuzzy probability model and extent return of the portfolio in the possibility model (Tanaka et al., 2000).

Furthermore, as compared to other models, the neuro-fuzzy model gives more accurate results under uncertainty. The neuro-fuzzy model is also applied to portfolio management. Basically, the Theory of fuzzy logic and fuzzy sets made neuro-fuzzy system. This system had the capacity to deal with non-linear and uncertain problems. Knowledge of human experts is utilized in attaining fuzzy reasoning with learning capabilities and noise robustness of neural networks for accurate results in neuro-fuzzy systems. The performance of the stock portfolio model is evaluated by its higher returns in terms of profits and risk taken(sekar, 2012)

The significant accomplishment of fuzzy logic in the field of control paved the way for its application in numerous different fields, including finance. Nonetheless, there has not been a refreshed and far-reaching writing survey on the applications of fuzzy logic in the budgetary field. Thus, this examination endeavors to basically analyze fuzzy logic as a helpful strategy to be applied to financial analysis and especially to the management of banking crises. The findings show that the fuzzy rationale has not yet been utilized to address banking emergencies or as a choice to guarantee the resolvability of banks while limiting the effect on the real economy (Sanchez-Roger et al., 2019).

For displaying of blended ceaseless discrete marvels, the fuzzy differential–difference equation is used. In the extraordinary case, the overall arrangement of straight fuzzy differential – contrast conditions are presented. The dynamical cycle in the spans is introduced by the comparison of fuzzy differential condition and with hasty hops in certain focuses. The evaluation is for the appropriateness of the model to examine the time estimation of cash (Long et al., 2020). A specialist framework that positions money related protections utilizing fuzzy participation capacities created and applied to shape portfolios. Outcomes show that the way to deal with structured stock portfolios can bring about higher returns than the market, as estimated by the profit for the S&P 500. These portfolios may likewise give predominant risk-adjusted returns when contrasted with the market (Sinha & Jacob, 2018).

Dark and Litterman propose another way to deal with gauge resource return. They present an approach to consolidate the financial specialist's perspectives into resource evaluating measure. Since the financial specialists see about future resource return consistently abstract and loose, by utilizing fuzzy numbers and the subsequent model that is multi–objective direct programming. Along these lines, the proposed model is investigated through a fuzzy trade-off programming approach utilizing fitting participation work. For this reason, the fuzzy ideal arrangement idea is dependent on the financial specialist inclination and lack of concern connections utilizing the authoritative portrayal of proposed fuzzy numbers by means of the corresponding cuts. A genuine world mathematical model is introduced in which MSCI (Morgan Stanley Capital Global Record) is picked as the objective list. The outcomes are accounted for a portfolio comprising of the six public files. The exhibitions of the proposed models are analyzed utilizing a few monetary standards (Gharakhani & Sadjadi, 2013).

From the literature review discussed above, it has been observed that fuzzy logic theory takes into account the uncertain and vague stock market information more efficiently and is a better measure when it comes to predicting future returns from past historical data (Ramli & Jaaman, 2017). Many researchers incorporate fuzzy theory for portfolio selection and optimization. Some took particular sector stock, a few a single country stock, others with a credibility approach, and the remaining with different risk measures (Gharakhani & Sadjadi, 2013; Glensk & Madlener, 2018; Liagkouras & Metaxiotis, 2018; Mohamed et al., 2010). Some comparison work has also been done where the conventional model is compared with the fuzzy model, taking into account different fuzzy approaches (Ramli & Jaaman, 2019).

DATA AND RESEARCH METHODOLOGY

Data Description

This study consists of monthly data of prices of the Indices of the economic bloc, i.e., BRICS (Brazil, Russia, India, China, and South Africa). Indices taken for BRICS are Brazil Bovespa (BVSP), South Africa (FTSE JSE Capped RAFI All Share Historical Data), China Shanghai Composite (SSEC), MOEX Russia Historical Data, India Nifty 500 Historical Data. Both of these economic blocs have emerging market economies. Basically a comparison is drawn using conventional approaches with fuzzy approaches for portfolio optimization. Lately, tracking error for efficient frontiers is calculated using the MSCI emerging market Index for the same period as above to see which portfolio has the lowest tracking error with the market index and hence is more efficient.

Methodology

The primary methodology used is the Fuzzy M-V Model using a Trapezoidal membership function for calculating Fuzzy Returns and Fuzzy Variance along with the Fuzzy Variance-Covariance Matrix. In order to give support to the former, Markowitz's M-V method is also used as a comparison, which is considered as one of the vital researches done in modern finance.

Mean-Variance(M-V) Model

It was proposed by Harry Markowitz in 1952, the aim is to maximize return for a given level of risk or to minimize risk for a given level of return, with the assumption of asset returns being normally distributed. M-V model is as follow:

$$\text{Min } v^2 = \sum\sum w_i.w_j.Cov_{i.j} \tag{1}$$

$$\text{s.t: } P_R \geq \Sigma\, w_i.R_i$$

$$\Sigma w_i = 1,\ wi \geq 0$$

Where P_R is the return of the portfolio subject to mean-variance, V^2 is portfolio risk, w_i is the proportion of investment in each security i, R_i is average rate of return for security i, $Cov_{i.j}$ is security i and j covariance in a portfolio and n is the number of securities in the portfolio

Fuzzy Mean-Variance (M-V) Model

In the fuzzy M-V model, the returns of securities are taken as fuzzy returns using a Trapezoidal fuzzy number for calculation, also variance and co-variance are computed using the same approach. Trapezoidal Fuzzy Number is defined in the form of parameters ranging from and including minimum, lower mode, upper mode and maximum (Ramli & Jaaman, 2019).

Suppose "A" is a trapezoidal fuzzy number with membership function μ(x), then;

$$\mu\,(x) = \begin{cases} \dfrac{x-a}{b-a}, & if\ a \leq x \leq b \\ 1, & if\ b \\ \dfrac{x-d}{c-d}, & if\ c \leq x \leq d \\ 0, & otherwise \end{cases} \tag{2}$$

With a ≤ b ≤ c ≤ d. It can be written as A = (a, b, c, d). For trapezoidal fuzzy number A, the α – level set is $[A]^\alpha = [a + \alpha\,(b\text{-}a),\ d - \alpha\,(d - c)]$.

Mean, Variance and Co-variance of the portfolio can be calculated using above Trapezoidal Fuzzy Number as mentioned below:

$$R\,(A) = \frac{1}{6}(a+d) + \frac{1}{3}\,(b+c) \tag{3}$$

$$V\,(A) = \frac{(a-d)^2}{4} + \frac{(c-d+a-b)^2}{8} - \frac{(d-a)(d-c+b-a)}{3} \tag{4}$$

$$Cov\ (Ai\ .\ Aj) = \frac{T1iT1j}{4} - \frac{T1iT2j+T2iT1J}{6} + \frac{T2iT2j}{8} \qquad (5)$$

With $T_{1k} = d_{k} - a_{k}$, $T_{2k} = d_{k} - c_{k} + b_{k} - a_{k}$, $k = i,j$

Combining (3), (4) and (5) for computing Portfolio Return and Variance;

$$Min\ v^2 = \sum_{i=1}^{n} w_i^2 \frac{(a-d)^2}{4} + \frac{(c-d+a-b)^2}{8} - \frac{(d-a)(d-c+b-a)}{3} + 2\sum_{i=1}^{n}\sum_{j=1}^{n} w_i . w_j \left(\frac{T1iT1j}{4} - \frac{T1iT2j+T2iT1J}{6} + \frac{T2iT2j}{8}\right) \qquad (6)$$

$$\text{s.t: } P_R \geq \Sigma\, wi\ \left(\frac{1}{6}(a+d) + \frac{1}{3}\ (b+c)\right)$$

$$\sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0$$

where P_R is portfolio return, V^2 is portfolio risk, a is the security i^{th} return at the 40^{th} and 5^{th} percentile, b is the security i^{th} return at the 40^{th} percentile, c is the security i^{th} return at the 60^{th} percentile, and d is the security i^{th} return at the 95^{th} and 60^{th} percentile.

Tracking Error

Tracking error of efficient frontier drawn from above fuzzy mean-variance method using trapezoidal fuzzy number is computed using following two tracking errors (Chu, 2011; Y. Fang & Wang, 2005; H. Zhang & Watada, 2018):

$$Avg.T.E = \sum_{t=1}^{T} \frac{1}{T}\ (R_t\ (x) - I_t) \qquad (7)$$

$$Standard\ Deviation\ T.E = sqrt(\sum_{t=1}^{T} \frac{1}{T}\ (R_t\ (x) - I_t)) \qquad (8)$$

Data Preprocessing- Fuzzy

All calculations are performed on excel. It consists of following steps:

- The monthly prices are converted to monthly return for each stock (country).
- Percentile technique is used for trapezoidal membership function. The 4 trapezoidal points of return for each stock are calculated using the Excel percentile formula on the 5^{th} (a), 40^{th} (b), 60^{th} (c), and 95^{th} (d) spread.
- The rest calculation is based on these 4 return points. Fuzzy average stock return is calculated using formula: $\frac{1}{6}(a+d) + \frac{1}{3}\ (b+c)$.
- Similarly, Fuzzy Variance is calculated breaking the formula as:
  - (a-d) * (a-d)
  - The sum in above cell divided by 4 -- i
  - (c-d+a-b) * (c-d+a-b)
  - The sum in immediate cell divided by 8 --------------------------------------ii
  - (d-a) (d-c+b-a)
  - The sum in immediate cell divided by 3---------------------------------------iii
  - Lastly, calculating the values in equation i, ii and iii as (i + ii – iii)
- The covariance is calculated in following steps.
  - T1 is calculated for each country’s stock through a - b

- T2 is calculated for each country's stock through d - c + b – a
- TThe values calculated in T1 and T2 are now used for calculation of each country's covariance with other country i.e. Brazil with South Africa, Brazil with China, Brazil with Russia, Brazil with India (one example). The formula for Cov (A_i, A_j) = $T1_i\ T1_j/4 - T1_i\ T2_j + T2_i\ T1_j/6 + T2_i\ T2_j/8$. Where $T1_k = d_k - a_k$ and $T2_k = d_k - c_k + b_k - a_k$, k = i,j

- Portfolio Mean, Variance, and Standard Deviation are calculated using the Excel formula = MMULT (fuzzy return rows, weight column), = MMULT (MMULT (TRANSPOSE (weight column), variance-covariance matrix), weight column), = SQRT (variance cell).
- For portfolio optimization and the efficient frontier curve, Excel Solver is used.
  - Setting the objective as maximizing the Portfolio Mean cell.
  - Changing Variable (weight column)
  - Constraints as the Sum of weights is equal to 1, and the Portfolio Variance cell is locked by using the F5 key, set equal to a series of variances. This is used to get optimized portfolio points for the efficient frontier curve.

The superiority is measured through portfolio return and risk. This is done by comparing the conventional M-V model with the fuzzy M-V model (Ramli & Jaaman, 2017). Two economic blocs are included in order to provide with more robust result. Two tracking errors are also used in order to check which of the superior portfolio is mimicking the benchmark index, adding more value to the study (Chu, 2011; Fang & Wang, 2005; Zhang & Watada, 2018).

Figures 1 to 5 show each country's return data which is normally distributed. For fuzzy returns calculation, this normal bell shape is broken down into a trapezoidal shape with 4 distinct points taken from the percentile formula.

Figure 1 shows monthly returns of Brazil- BRICS.

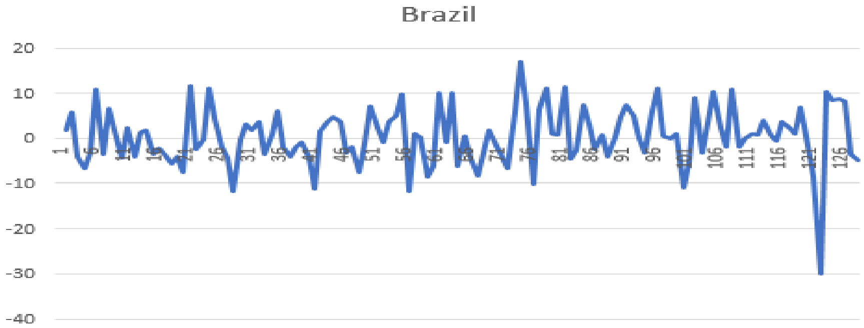


Figure 2 shows monthly returns of China- BRICS.

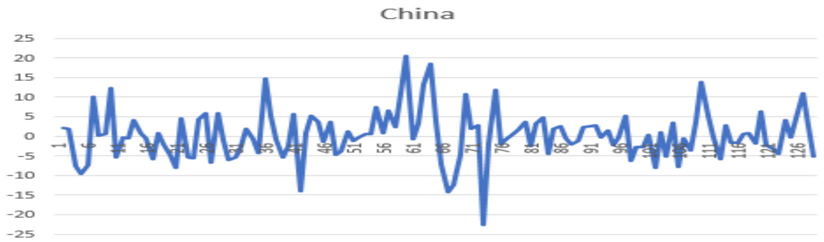


Figure 3 shows monthly returns of South Africa- BRICS.

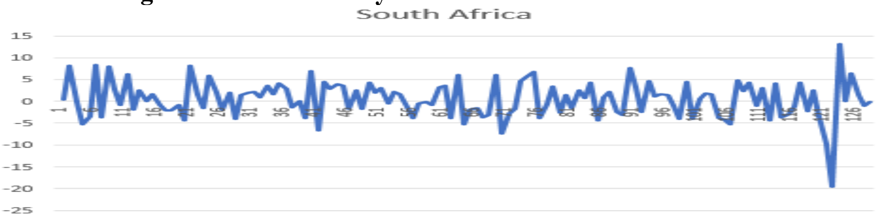
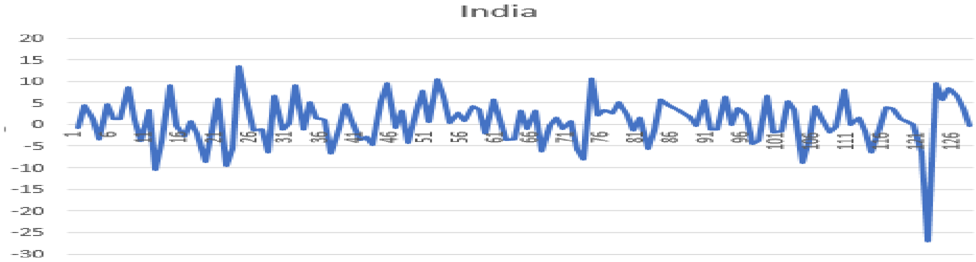


Figure 4 shows monthly returns of Russia- BRICS.

Figure 5 shows monthly returns of India- BRICS.



RESULTS AND ANALYSIS

Table 1 shows the summary statistics of BRICS. It consists of Mean, Median, Standard Deviation, Kurtosis, and Skewness. Keeping the study restricted to Mean and Standard Deviation only, it shows that the highest mean is of India (0.77), followed by Russia (0.67), South Africa (0.46), Brazil (0.50), and China (0.25), respectively. Taking into account the risk (Standard Deviation), the country with the highest risk is Brazil, having 6.36, and it is fourth when it comes to return. The remaining sequence from highest to lowest risk continues like this: China > India > Russia > South Africa.

Table 1: Summary Statistics for BRICS, showing maximum return and maximum risk for BRICS countries

	BRAZIL	SOUTH AFRICA	CHINA	RUSSIA	INDIA
Mean	0.50	0.46	0.25	0.67	0.77
Standard Error	0.56	0.37	0.55	0.42	0.46
Median	0.56	0.66	-0.11	1.05	0.86
Standard Deviation	6.37	4.22	6.21	4.76	5.20
Sample Variance	40.52	17.77	38.53	22.68	27.07
Kurtosis	3.15	3.32	2.08	0.94	5.40
Skewness	-0.63	-0.60	0.16	-0.09	-1.12
Count	128	128	128	128	128

Portfolio Composition

Comparison of M-V Model with Fuzzy M-V Model- BRICS

Table 2 indicates the portfolio composition of the conventional M-V model and with fuzzy M-V model. Weights with non-zero value show that the shares will be purchased from these countries, while a zero value indicates no share buying in order to form an optimum portfolio. The value or quantitative number that gives the optimum value for a certain model is known as the optimum portfolio composition. The M-V model allocates weight in Russia and India, such as 4.57% and 95.42% respectively. Whereas, employing the Fuzzy M-V method investment proportion is 12.15% and 87.84% for South Africa and India, respectively.

Table 2: Showing weights invested in each security of BRICS for M-V and Fuzzy M-V method

Stocks	Mean-Variance Method (%)	Fuzzy Mean-Variance Method (%)
BRAZIL	0	0
SOUTH AFRICA	0	12.15
CHINA	0	0
RUSSIA	4.57	0
INDIA	95.42	87.85

Portfolio Performance

Equally Weighted Portfolios – BRICS Case

Table 3 shows the comparison of the M-V model with the Fuzzy M-V model employing Trapezoidal Fuzzy Return and Variance for computational purposes. The results in the table are in support of the study, i.e., Fuzzy M-V e is a more efficient model when it comes to dealing with uncertain market conditions. The findings show that Fuzzy M-V gives high portfolio returns (0.64) compared to the conventional M-V Model with (0.53) portfolio returns. Also, the Fuzzy M-V model is less risky with (3.83) as compared to the conventional M-V Model with a high portfolio risk of (3.90). To get a clearer picture portfolio-performance ratio of return with risk is calculated. As per that ratio Fuzzy M-V model is a better performing model with high return and less risk (both) as compared to the conventional M-V model.

Table 3: M-V and Fuzzy M-V Portfolio Return and Risk for equally weighted portfolios- BRICS

	M-V	FUZZY M-V
MEAN (Return)	0.53	0.64
S.D. (Risk)	3.90	3.83
Portfolio-Performance (Mean/S.D)	0.14	0.17

Optimized Portfolios- BRICS Case

Portfolio optimization refers to the situation with unequal weight assumption (weights allocated on the basis of highly performing securities). Table 4 comprises the data when both the M-V and Fuzzy M-V models are optimized to get a high return on less risk. This optimization case also supports the study, as the given stats show more returns for the Fuzzy M-V Model, which is 0.84, as compared to the conventional M-V model with a return of 0.77. Similarly, the risk associated with both models is in support of the Fuzzy M-V model with less risk of (3.52) compared to the M-V model with more risk of (5.04). To get a clearer picture, portfolio performance is also calculated by using the ratio of return to risk. As per that ratio Fuzzy M-V model is a better performing model with both (high return and less risk) as compared to the conventional M-V model, with given stats of 0.24 and 0.15, respectively.

Table 4: M-V and Fuzzy M-V Portfolio Return and Risk for optimized portfolios- BRICS

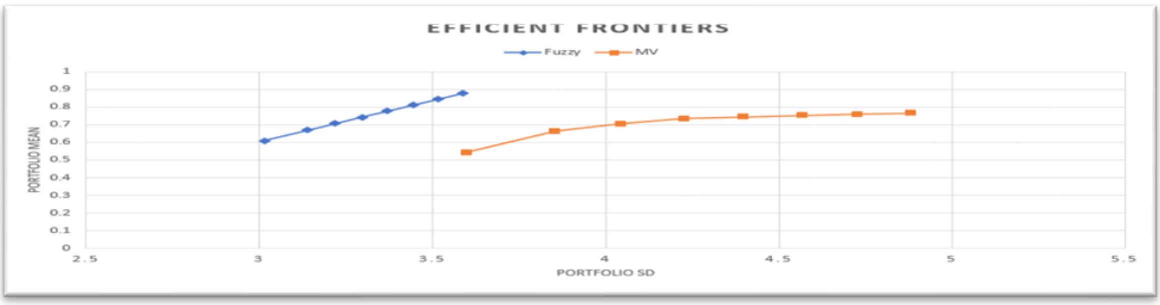
	M-V	FUZZY M-V
MEAN (Return)	0.77	0.84
S.D. (Risk)	5.04	3.52
Portfolio-Performance (Mean/S.D)	0.15	0.24

Efficient Frontier

Combined Efficient Frontier for BRICS:

Figure 6 shows the combined efficient frontier curves of M-V and Fuzzy M-V. It is evident from the graph that the Fuzzy M-V model provides a better and superior performing portfolio. As the efficient frontier curve is more towards the y-axis, i.e., comprises of portfolio of securities with more returns and less risk. Whereas, the conventional M-V efficient frontier is more towards the North-East, comprising of portfolio of securities with low returns and high risk.

Figure 6: Shows combined efficient frontiers (M-V and Fuzzy M-V) for BRICS



Tracking Error:

Table 5 shows the tracking error for BRICS using the average tracking error and the standard deviation of tracking error. The average tracking error for BRICS is 0.76. Also, the standard deviation tracking error for BRICS is 0.573008.

Table 5: Average Tracking Error and Standard Deviation Tracking Error for BRICS

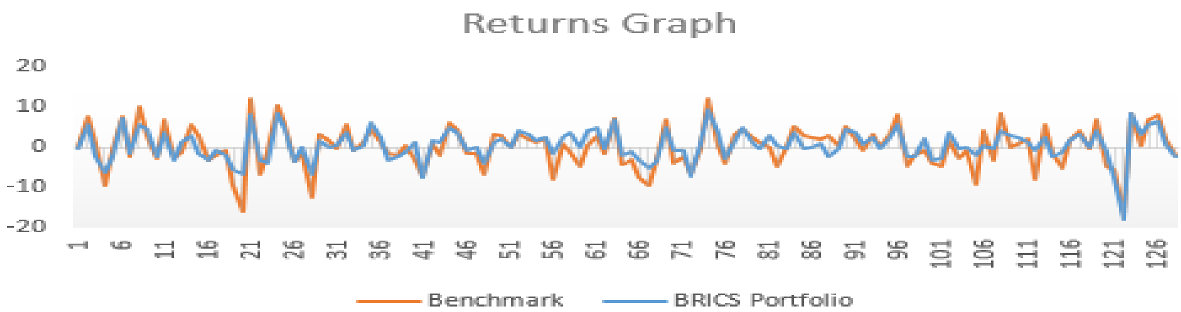
	Average Tracking Error	S.D Tracking Error
BRICS	0.76	0.57

Figure 7 shows the return graph trend for the BRICS Portfolio and the Benchmark MSCI emerging market Index. The portfolio line closely follows the benchmark line.

Figure 7: Shows returns for BRICS and Benchmark Index in the case of Fuzzy M-V

Figure 8 shows returns for BRICS and the Benchmark Index in the case of the M-V framework. The portfolio line closely follows the benchmark line.

Figure 8: Shows returns for BRICS and the Benchmark Index in the case of M-V



The above figures show each country's return data, which is normally distributed and is towards the mean line. For fuzzy returns calculation, this normal bell shape is broken down into a trapezoidal shape with 4 distinct points taken from the percentile formula. Table 6 is presented below:

Table 6: Trapezoidal Returns Value- BRICS

	BRAZIL	S. AFRICA	CHINA	RUSSIA	INDIA
5 percentile	-8.55246	-4.96974	-7.92011	-7.96622	-6.53005
40 percentile	-1.24049	-0.80776	-0.82929	0.01925	-0.25746
60 percentile	1.64733	1.669098	1.042633	1.782414	1.753655
95 percentile	10.80987	6.903597	11.4551	7.602749	8.797183

DISCUSSION

The study is performed and supported by comparing the conventional M-V model with the new Fuzzy M-V model, specifically using trapezoidal fuzzy numbers for the computation of mean, variance, and covariance. This comparison has proven the Fuzzy M-V model a better measure at various points, as shown in the above section of results and interpretation, and how it is helpful for investors when it comes to investment strategy.

For equally weighted portfolios, BRICS is getting a higher return and lower risk of the portfolio with the Fuzzy M-V model proving that the said model is a better measure dealing with uncertain stock market conditions as compared to the conventional M-V model, because for same returns, conventional M-V model has less return and more risk. In order to provide more consistent results same comparison is drawn for the "Optimized Portfolio" of the BRICS bloc, and the results are no different from the former equally weighted BRICS bloc. Also, supporting the findings of the studies of Mohamed et al.(2010) and Ramli & Jaaman (2017).

Efficient frontier refers to the combination of portfolios optimized in a way that either gives the highest return for a given level of risk or the lowest risk for a given level of return. The graphs presented in the results section also get support from the literature. The comparison also shows that the conventional M-V efficient frontier for the BRICS case, the efficient frontier of the Fuzzy M-V model is more towards the North-west axis (y-axis) (Mohamed et al., 2010; Sinha & Jacob, 2018).

Portfolio composition basically shows how much investment is made in each security for the "Optimal Portfolio" case. It seems a huge difference for BRICS when the economic blocs are tested for M-V and Fuzzy M-V model individually. Referring to BRICS major proportion is invested in India (87.8%), and the rest (12.15%) in South Africa, while other countries with zero investment in the Fuzzy M-V model. It shows the importance of the right model to be employed for practicality and accurate results (Allen et al., 2003; Ramli & Jaaman, 2019).

At the end, tracking error for the superior portfolios is also computed, i.e., the Fuzzy M-V model for BRICS. Two different tracking errors are calculated for BRICS using the MSCI emerging market index as a benchmark. One is defined as the returns of securities and the benchmark index return. In simpler terms, it is a measure which investors can see the extent to which returns of underlying securities vary from the benchmark (Chu, 2011) and the second tracking error used is standard deviation of difference between returns of each portfolio with the benchmark index (Chu, 2011; Fang & Wang, 2005).

The graphical representation also shows the deviation of return graphs of BRICS with the MSCI emerging market index, which is serving as a benchmark index in this study (H. Zhang & Watada, 2018). The results are consistent with past studies and follow a similar pattern, confirming that Fuzzy M-V is a better measure for dealing with uncertain stock market conditions (Ramli & Jaaman, 2019; Ramli & Jaaman, 2017).

CONCLUSION

This study proposes, explains, and justifies how the technique of fuzzy when employed for portfolio optimization, gives better results. The overall results are compared with the conventional model of M-V to confirm the real picture. In other words, it could be said that the conventional M-V model is extended into a fuzzy M-V model specifically using trapezoidal fuzzy numbers to account for the computation of fuzzy return (mean), risk (variance), and co-variance (Mohamed et al., 2010). The monthly data, i.e., from 1st January 2010 till 31st Sep 2020 of BRICS are taken into account for computational the conventional M-V method and new Fuzzy M-V method are computed with the same data. Next, tracking error is computed for the efficient method, as confirmed in the prior step, which is the Fuzzy M-V model with its benchmark MSCI emerging market index.

The general findings of the returns for BRICS show that India is getting the highest return, whereas high risk is associated with Brazil. The results of the Fuzzy M-V model and conventional M-V model for the "equal weighted" assumption do not giving consistent results. BRICS has having higher portfolio return and less portfolio risk in the case of the Fuzzy M-V model. Whereas, the results for "optimized portfolio" fully support the proposed study for BRICS. High portfolio return and low portfolio risk are associated Fuzzy M-V model, proving the efficiency and superiority of the model in dealing with uncertain stock market conditions in a better way (Sinha & Jacob, 2018). It shows that the Fuzzy technique works best for the optimized portfolio case.

The findings also show that optimized portfolios for BRICS portfolio composition differ in both the conventional M-V model and the Fuzzy M-V model. This investment proportion matters a lot when it comes to investment decisions, and the study confirms that the Fuzzy M-V model is giving the best results as compared to the conventional method (Shang & Hossen, 2013). For BRICS in the case of fuzzy, a major proportion of investment needs to be done in India, which is 95.42% followed by South Africa, which is 4.57%. Efficient frontier curves for both models also support the findings of the study (Mohamed et al., 2010).

The results can be summarized as, for uncertain stock market conditions Fuzzy M-V model doesn't provide consistent results for "equally weighted portfolios". The Fuzzy M-V model works best, is consistent, and provides high return and low risk for the "optimized portfolio". Average tracking error and Standard Deviation of tracking error are also computed and provide consistent results for the Fuzzy M-V model. This study gives recommendations and policy implications for investors and policymakers of the BRICS bloc, who, while making investment decisions, sense an uncertain environment by using fuzzy optimization. As these countries have high economic growth to be expected in the coming years and is seem to be an attractive destination for investors. So this is also an important contribution of the study for the policymakers. Furthermore, it is recommended for researchers and academicians on how they can obtain the optimal solution for portfolio allocation, i.e., getting the high profits with less or equal risk. This can be taken into account by both risk-takers and risk-averse investors. The future directions of this study are that the same study can be performed taking into account the fuzzy membership function for investors' aspiration level, i.e., knowledge of experts and investors' personal opinion, subject to expected return and risk. More work can be done employing different risk measures and then comparing and proving the efficiency of the model with the model proposed in this study.

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